

HS_r -valued Gauss maps and umbilic spacelike surfaces of codimension two

Dang Van Cuong and Doan The Hieu*

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Abstract

To study spacelike surfaces of codimension two in the Lorentz-Minkowski space \mathbb{R}_1^{n+1} , we construct a pair of maps whose values are in $HS_r := H_+^n(\mathbf{v}, 1) \cap \{x_{n+1} = r\}$, called \mathbf{n}_r^\pm -Gauss maps. It is showed that they are well-defined and useful to study practically flat as well as umbilic spacelike surfaces of codimension two in \mathbb{R}_1^{n+1} .

1 Introduction

In classical differential geometry, the Gauss map plays an important role in the study of the behaviour or geometric invariants of surfaces of codimension one. In the case of surfaces of codimension larger than one, Gauss map associated with some arbitrary normal field ν is considered. By that way, one can consider the second fundamental form associated with ν and study invariants or properties of surfaces, concerning to the concept of ν -curvatures, that are dependent or independent on ν .

In 1989, Marek Kossowski [8] used Gauss maps, whose values are in the lightcone, to study spacelike 2-surfaces in \mathbb{R}_1^4 , followed by Izumiya et. al. (see [3]). In 2004, Izumiya et. al. [5] used Gauss maps associated with a normal field ν to study ν -umbilicity for spacelike surfaces of codimension two in Lorentz-Minkowski spaces. Long before, in the

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study of minimal 2-surfaces in \mathbb{R}^n , it is well-known that the mean curvature vector \vec{H} does not depend on ν (see [10]).

Motivated by these ideas, to study practically spacelike surfaces of codimension two in \mathbb{R}_1^{n+1} , we construct a kind of Gauss map whose values are in a hyperbolic space, called \mathbf{n}_r^\pm -Gauss maps.

Let M be a spacelike surface of codimension two in \mathbb{R}_1^{n+1} . The normal plane of M at $p \in M$, denoted by $N_p M$ is a timelike 2-plane. We identify $N_p M$ with its image under the translation given by the vector $-p$. Then, the intersection of $N_p M$ and the hyperbolic space with center $\mathbf{v} = (0, 0, \dots, 0, -1)$ and radius 1, $H_+^n(\mathbf{v}, 1)$, is a hyperbola. For a fixed $r > 0$, the hyperplane $\{x_{n+1} = r\}$ meets this hyperbola exactly at two points, denoted by $\mathbf{n}_r^\pm(p)$.

This gives two differential maps $p \mapsto \mathbf{n}_r^\pm(p)$, called \mathbf{n}_r^\pm -Gauss maps. Their derivatives are self-adjoint, and hence we can define the \mathbf{n}_r^\pm -Weingarten maps, \mathbf{n}_r^\pm -Gauss-Kronecker curvatures, \mathbf{n}_r^\pm -mean curvatures, \mathbf{n}_r^\pm -principal curvatures, \mathbf{n}_r^\pm -flat points, \mathbf{n}_r^\pm -umbilic points \dots .

We use these maps to study the flatness and umbilicity for spacelike surfaces of codimension two in \mathbb{R}_1^{n+1} .

In this situation, some criteria for a spacelike surface to be flat or umbilic as well as examples of some kinds of flat and umbilic spacelike surfaces of codimension two are established. These examples show that we can use \mathbf{n}_r^\pm -Gauss maps to study some properties of spacelike surfaces of codimension two practically.

2 Preliminaries

2.1 The Lorentz-Minkowski space \mathbb{R}_1^{n+1}

The Lorentz-Minkowski space \mathbb{R}_1^{n+1} is the $(n+1)$ -dimensional vector space $\mathbb{R}^{n+1} = \{(x_1, x_2, \dots, x_{n+1}) : x_i \in \mathbb{R}, i = 1, 2, \dots, n+1\}$ endowed the pseudo scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i - x_{n+1} y_{n+1},$$

where $\mathbf{x} = (x_1, x_2, \dots, x_{n+1})$, $\mathbf{y} = (y_1, y_2, \dots, y_{n+1}) \in \mathbb{R}^{n+1}$. Since $\langle \cdot, \cdot \rangle$ is non-positive definite, $\langle \mathbf{x}, \mathbf{x} \rangle$ may be zero or negative. We say a nonzero vector $\mathbf{x} \in \mathbb{R}_1^{n+1}$ spacelike, lightlike or timelike if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$, respectively. If $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, we say \mathbf{x}, \mathbf{y} are pseudo-orthogonal.

The norm of a vector $\mathbf{x} \in \mathbb{R}_1^{n+1}$, denoted by $\|\mathbf{x}\|$, is defined by $\sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. For a nonzero vector $\mathbf{n} \in \mathbb{R}_1^{n+1}$, a hyperplane with the pseudo normal \mathbf{n} is defined as

$$HP(\mathbf{n}, c) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} : \langle \mathbf{x}, \mathbf{n} \rangle = c, c \in \mathbb{R}\}.$$

The hyperplane is said to be spacelike, lightlike or timelike if \mathbf{n} is timelike, lightlike or spacelike, respectively.

It is easy to see that, $HP(\mathbf{n}, c)$ is spacelike if any vector $\mathbf{x} \in HP(\mathbf{n}, 0)$ is spacelike; $HP(\mathbf{n}, c)$ is lightlike if $HP(\mathbf{n}, 0)$ is tangent to the lightcone and $HP(\mathbf{n}, c)$ is timelike if $HP(\mathbf{n}, 0)$ contains timelike vectors.

In \mathbb{R}_1^{n+1} , we have three kinds of pseudo-hyperspheres

1. $H^n(\mathbf{a}, R) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = -R^2, R > 0\}$: the hyperbolic with center \mathbf{a} and radius R ;
2. $S_1^n(\mathbf{a}, R) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = R^2, R > 0\}$: the de Sitter with center \mathbf{a} and radius R ;
3. $LC(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = 0\}$: the lightcone with vertex \mathbf{a} .

And we call

$$H_+^n(\mathbf{a}, R) = \{\mathbf{x} \in H^n(\mathbf{a}, R) : x_{n+1} - a_{n+1} \geq 0\}$$

the hyperbolic space with center \mathbf{a} and radius R .

2.2 The \mathbf{n}_r^\pm -Gauss maps

In this paper a surface is always spacelike and is of codimension two in \mathbb{R}_1^{n+1} , unless otherwise stated. It is an embedding $\mathbf{X} : U \rightarrow \mathbb{R}_1^{n+1}$, where U is an open domain in \mathbb{R}^{n-1} . We often identify $M = \mathbf{X}(U)$ with \mathbf{X} .

In this section we introduce two concrete spacelike normal fields on a surface that are useful to study the flatness and umbilicity, practically.

The normal plane of M at $p \in M$, denoted by $N_p M$, can be viewed as a timelike 2-plane passing the origin. The intersection of this plane and the hyperbolic space with center $\mathbf{v} = (0, 0, \dots, 0, -1)$ and radius 1, $H_+^n(\mathbf{v}, 1)$ is a hyperbola. For a fixed $r > 0$, the hyperplane $\{x_{n+1} = r\}$ meets this hyperbola exactly at two points, denoted by $\mathbf{n}_r^\pm(p)$.

DEFINITION 2.1. The following maps

$$\begin{aligned} \mathbf{n}_r^\pm : M &\rightarrow HS_r := H_+^n(\mathbf{v}, 1) \cap \{x_{n+1} = r\} \\ p &\mapsto \mathbf{n}_r^\pm(p). \end{aligned}$$

are called \mathbf{n}_r^\pm -Gauss maps.

The first property of \mathbf{n}_r^\pm -Gauss maps is

THEOREM 2.2. *The \mathbf{n}_r^\pm -Gauss maps are smooth.*

PROOF. Locally, $\mathbf{n}_r^\pm(p)$ are the solutions of the following system of equations

$$\begin{cases} \langle \mathbf{X}_{u_i}, \mathbf{a} \rangle &= 0, \quad i = 1, 2, \dots, n-1; \\ \langle \mathbf{a} - \mathbf{v}, \mathbf{a} - \mathbf{v} \rangle &= -1; \end{cases}$$

where $\mathbf{a} = (a_1, a_2, \dots, a_n, r)$.

Since $\text{rank}(\mathbf{X}_{u_1}, \mathbf{X}_{u_2}, \dots, \mathbf{X}_{u_{n-1}}) = n-1$, we can assume that a_1, a_2, \dots, a_{n-1} are linearly expressed in term of a_n . Substituting these to the last equation, we get a quadratic equation in term of a_n . This equation has exactly two solutions and of course they are smooth. \square

From now on, the symbol “ $*$ ” means “ $+$ ” or “ $-$ ”, unless otherwise stated.

The derivative of \mathbf{n}_r^* at p

$$d\mathbf{n}_r^*(p) : T_p M \rightarrow T_{\mathbf{n}_r^*(p)} H_+^n(\mathbf{v}, 1) \subset T_p M \oplus N_p M;$$

can be written as

$$d\mathbf{n}_r^*(p) = d\mathbf{n}_r^{*T}(p) + d\mathbf{n}_r^{*N}(p),$$

where $d\mathbf{n}_r^{*T}$ and $d\mathbf{n}_r^{*N}$ are the tangent and normal components of $d\mathbf{n}_r^*$, respectively.

We recall some definitions and facts concerning to ν -umbilic (see [5]) but restated for \mathbf{n}_r^* . Denoted by

1. $A_p^{\mathbf{n}_r^*} := -d\mathbf{n}_r^{*T}(p)$, the \mathbf{n}_r^* -Weingarten map of M at p ;
2. $K_p^{\mathbf{n}_r^*} := \det(A_p^{\mathbf{n}_r^*})$, the \mathbf{n}_r^* -Gauss-Kronecker curvature of M at p ;
3. $H_p^{\mathbf{n}_r^*} := \frac{1}{n-1} \text{tr}(A_p^{\mathbf{n}_r^*})$, the \mathbf{n}_r^* -mean curvature of M at p ;
4. $k_1^{\mathbf{n}_r^*}(p), k_2^{\mathbf{n}_r^*}(p), \dots, k_{n-1}^{\mathbf{n}_r^*}(p)$, (the eigenvalues of $A_p^{\mathbf{n}_r^*}$) the \mathbf{n}_r^* -principal curvatures of M at p .

Of course

$$K_p^{\mathbf{n}_r^*} = k_1^{\mathbf{n}_r^*}(p) k_2^{\mathbf{n}_r^*}(p) \dots k_{n-1}^{\mathbf{n}_r^*}(p),$$

and

$$H_p^{\mathbf{n}_r^*} = \frac{1}{n-1} (k_1^{\mathbf{n}_r^*}(p) + k_2^{\mathbf{n}_r^*}(p) + \dots + k_{n-1}^{\mathbf{n}_r^*}(p)).$$

We have some well-known facts.

1. The \mathbf{n}_r^* -Weingarten map is self-adjoint.

2. The \mathbf{n}_r^* -principal curvatures $k_i^{\mathbf{n}_r^*}(p), i = 1, 2, \dots, n-1$ of M at p are the solutions of the following equation

$$(1) \quad \det(b_{ij}^{\mathbf{n}_r^*}(p) - kg_{ij}(p)) = 0,$$

where $b_{ij}^{\mathbf{n}_r^*}(p) := \langle \mathbf{X}_{u_i u_j}(p), \mathbf{n}_r^*(p) \rangle$, $i, j = 1, 2, \dots, n-1$, the coefficients of the \mathbf{n}_r^* -second fundamental form of M at p .

3. $K_p^{\mathbf{n}_r^*} = \det(b_{ij}^{\mathbf{n}_r^*}(p)) \cdot \det(g_{ij}(p))^{-1}$.

DEFINITION 2.3. 1. A point $p \in M$ is said to be \mathbf{n}_r^* -umbilic if $k_i^{\mathbf{n}_r^*}(p) = k(p)$, $i = 1, 2, \dots, n-1$. If $k(p) = 0$, then p is called \mathbf{n}_r^* -flat.

2. M is said to be \mathbf{n}_r^* -umbilic (\mathbf{n}_r^* -flat) if every point $p \in M$ is \mathbf{n}_r^* -umbilic (\mathbf{n}_r^* -flat).
3. M is said to be totally umbilic (totally flat) if every point $p \in M$ is \mathbf{n}_r^* -umbilic (\mathbf{n}_r^* -flat) for every $r > 0$.

3 The \mathbf{n}_r^* - flatness

We begin with a useful lemma.

LEMMA 3.1. If $(\mathbf{n}_r^*)_{u_i} \in N_p M$, where $i \in \{1, 2, \dots, n-1\}$, then $(\mathbf{n}_r^*)_{u_i} = 0$.

PROOF. We observe that, the last coordinate of $(\mathbf{n}_r^*)_{u_i}$ is zero because the last coordinate of \mathbf{n}_r^* is constant. Therefore, since $\{\mathbf{n}_r^+, \mathbf{n}_r^-\}$ is a basis of $N_p M$, we have

$$(2) \quad (\mathbf{n}_r^*)_{u_i} = \lambda(\mathbf{n}_r^+ - \mathbf{n}_r^-).$$

An easy calculation shows that $\langle \mathbf{n}_r^*, \mathbf{n}_r^* \rangle = 2r$. Therefore,

$$\langle (\mathbf{n}_r^*)_{u_i}, \mathbf{n}_r^* \rangle = \lambda \langle \mathbf{n}_r^+ - \mathbf{n}_r^-, \mathbf{n}_r^* \rangle = 0.$$

If $\lambda \neq 0$, then

$$\langle \mathbf{n}_r^+, \mathbf{n}_r^+ \rangle = \langle \mathbf{n}_r^-, \mathbf{n}_r^- \rangle = \langle \mathbf{n}_r^+, \mathbf{n}_r^- \rangle = 2r.$$

And hence,

$$\langle \mathbf{n}_r^+ - \mathbf{n}_r^-, \mathbf{n}_r^+ - \mathbf{n}_r^- \rangle = 0,$$

a contradiction, because $\mathbf{n}_r^+ - \mathbf{n}_r^-$ is a nonzero spacelike vector. Thus, $\lambda = 0$, and the lemma is proved. \square

THEOREM 3.2. Let M be a connected surface. The following statements are equivalent

1. there exists an $r > 0$, M is \mathbf{n}_r^* -flat;

2. there exists an $r > 0$, \mathbf{n}_r^* is constant;
3. there exists a spacelike vector $\mathbf{a} = (a_1, a_2, \dots, a_n, a_{n+1})$, $a_{n+1} \neq 0$ and a real number c such that $M \subset HP(\mathbf{a}, c)$.

PROOF. (1. \Rightarrow 2.) Since M is \mathbf{n}_r^* -flat, i.e. $A_p^{\mathbf{n}_r^*} = 0$, we have

$$(3) \quad \langle \mathbf{X}_{u_i u_j}, \mathbf{n}_r^* \rangle = -\langle \mathbf{X}_{u_i}, (\mathbf{n}_r^*)_{u_j} \rangle = 0, \quad i, j = 1, 2, \dots, n-1.$$

But (3) means that $(\mathbf{n}_r^*)_{u_i} \in N_p M$ and hence $(\mathbf{n}_r^*)_{u_i} = 0$, $i = 1, 2, \dots, n-1$ by virtue of Lemma 3.1.

(2 \Rightarrow 1) Obviously.

(2. \Rightarrow 3.) If \mathbf{n}_r^* is constant, then

$$\frac{\partial}{\partial u_i} \langle \mathbf{X}, \mathbf{n}_r^* \rangle = \langle \mathbf{X}_{u_i}, \mathbf{n}_r^* \rangle - \langle \mathbf{X}, (\mathbf{n}_r^*)_{u_i} \rangle = 0.$$

Thus $\mathbf{X} \subset H(\mathbf{n}_r^*, c)$, for some constant c .

(3. \Rightarrow 2.) If M is contained in a timelike hyperplane with a unit spacelike normal vector $\mathbf{a} = (a_1, a_2, \dots, a_n, a_{n+1})$, $a_{n+1} \neq 0$, then it is not hard to check that we can choose the constant vector $\mathbf{n}_r^* = 2a_{n+1}\mathbf{a} \in H_+^n(\mathbf{v}, 1)$. \square

REMARK 3.3. 1. The Theorem 3.2 is a necessary and sufficient condition for a surface belonging to a timelike hyperplane that does not contain the x_{n+1} -axis. For the case of surfaces belonging to a timelike hypersurface containing the x_{n+1} -axis, see Example 5.2.

2. A necessary and sufficient condition for a surface belonging to a lightlike hyperplane based on the totally lightlike flatness was established in [6].
3. A similar result with an assumption on parallelism of the normal field was given ([5, Theorem 4.3]).

COROLLARY 3.4. *Let M be a connected surface and $\mathbf{n}_{r_1}^* \neq \mathbf{n}_{r_2}^*$, i.e. $r_1 \neq r_2$ or $\mathbf{n}_{r_1}^* = \mathbf{n}_r^+$, $\mathbf{n}_{r_2}^* = \mathbf{n}_r^-$ for some fixed r . If M is both $\mathbf{n}_{r_1}^*$ - and $\mathbf{n}_{r_2}^*$ -flat, then M is a part of a spacelike $(n-1)$ -plane. In this cases, \mathbf{n}_r^* are constant for every $r > 0$ or equivalently, M is totally flat, i.e. \mathbf{n}_r^* -flat for every $r > 0$.*

COROLLARY 3.5. *If M is connected and contained in a timelike hyperplane not containing the x_{n+1} -axis, then there exists a unique positive real number r such that M is \mathbf{n}_r^* -flat unless M is (or a part of) a spacelike $(n-1)$ -plane.*

4 The \mathbf{n}_r^* -umbilicity

In this section, we study the \mathbf{n}_r^* -umbilicity for spacelike surfaces of codimension two in \mathbb{R}_1^{n+1} . For a pseudo-hypersphere, we mean a hyperbolic or a de Sitter with center \mathbf{a} and radius R , or a lightcone with vertex \mathbf{a} . Because \mathbf{n}_r^\pm -umbilicity is an invariant under translations, we can assume that \mathbf{a} is the origin in the study of the \mathbf{n}_r^* -umbilicity for surfaces lying in a pseudo-hypersphere. We begin this section with another useful lemma.

LEMMA 4.1. *Suppose that ν_1 and ν_2 are smooth normal fields on M and for every $p \in M$, $\nu_1(p), \nu_2(p)$ are linear independent. If M is both ν_1 - and ν_2 -umbilic then M is ν -umbilic for every smooth normal field ν .*

PROOF. By the assumption, for every smooth normal field ν

$$\nu = \lambda_1 \nu_1 + \lambda_2 \nu_2$$

where λ_i , $i = 1, 2$ are smooth functions on M .

Because $d(\lambda_i \nu_i)^T = \lambda_i d(\nu_i)^T$, $i = 1, 2$

$$A^\nu = \lambda_1 A^{\nu_1} + \lambda_2 A^{\nu_2}.$$

Since $A^{\nu_i} = k^{\nu_i} \text{id}$, $i = 1, 2$

$$A^\nu = (\lambda_1 k^{\nu_1} + \lambda_2 k^{\nu_2}) \text{id}.$$

□

Because \mathbf{n}_r^+ , \mathbf{n}_r^- are linear independent by the construction and so are $\mathbf{n}_{r_1}^*$, $\mathbf{n}_{r_2}^*$ if $r_1 \neq r_2$, we have

COROLLARY 4.2. *If M is $\mathbf{n}_{r_1}^*$ - and $\mathbf{n}_{r_2}^*$ -umbilic, where $\mathbf{n}_{r_1}^* \neq \mathbf{n}_{r_2}^*$; then M is totally umbilic.*

REMARK 4.3. 1. By virtue of Lemma 4.1, a surface is totally umbilic iff it is ν -umbilic for every smooth normal field ν .

2. It is well-known that (see [5, Lemma 4.1]), a surface lying in a pseudo-hypersphere is always ν -umbilic, where ν is the position vector field. Therefore, Lemma 4.1 is useful in the study of the totally umbilicity for surfaces lying in a hyperbolic or a lightcone, because the position vector field and \mathbf{n}_r^* are always linear independent. The case of the de Sitter can be studied in a similar way by using the lightcone Gauss maps (see [3], [8]...). So for simplicity in statements, we just state for the case of the hyperbolic spaces.

By using of Theorem 3.2, Lemma 4.1 or by a direct computation (see Example 5.2), we have

COROLLARY 4.4. *If M is contained in the intersection of a hyperbolic space and a hyperplane, then M is totally umbilic.*

THEOREM 4.5. *Let M be a spacelike surface of codimension two in $H_+^n(0, R)$. The following statements are equivalent.*

1. *there exists $r > 0$, M is \mathbf{n}_r^* -umbilic;*
2. *M is totally umbilic;*
3. *M is contained in a hyperplane.*

PROOF. (1. \Rightarrow 2.) Because M is contained in $H_+^n(0, R)$, M is umbilic with respect to the position vector field \mathbf{X} . Moreover, because \mathbf{X} is timelike while \mathbf{n}_r^* is spacelike, M is totally umbilic by virtue of Lemma 4.1.

(2. \Rightarrow 3.) Let

$$\nu = \frac{\mathbf{X} \wedge \mathbf{X}_{u_1} \wedge \mathbf{X}_{u_2} \wedge \cdots \wedge \mathbf{X}_{u_{n-1}}}{|\mathbf{X} \wedge \mathbf{X}_{u_1} \wedge \mathbf{X}_{u_2} \wedge \cdots \wedge \mathbf{X}_{u_{n-1}}|}.$$

Because

$$(4) \quad \langle \nu, \mathbf{X} \rangle = 0, \quad \langle \nu, \nu \rangle = \pm 1, \quad \langle \nu, \mathbf{X}_{u_i} \rangle = 0, \quad i = 1, 2, \dots, n-1;$$

we have

$$\langle d\nu, \mathbf{X} \rangle = \langle \nu, d\mathbf{X} \rangle = 0; \quad \langle \nu, d\nu \rangle = 0.$$

Since $\{\nu, \mathbf{X}\}$ is a basis of $N_p M$, $d\nu \in T_p M$, i.e. ν is parallel.

By virtue of Lemma 4.2 in [5], $d\nu = \lambda d\mathbf{X}$, where λ is constant and hence $\nu = \lambda \mathbf{X} + \mathbf{a}$, where \mathbf{a} is a constant vector. Since $\langle \nu, \mathbf{X} \rangle = 0$, $\langle \mathbf{X}, \mathbf{a} \rangle = -\langle \mathbf{X}, \lambda \mathbf{X} \rangle = -\lambda R = c$ (a constant). Thus, $M \subset HP(\mathbf{a}, c)$.

(3. \Rightarrow 1.) follows by Corollary 4.4. □

LEMMA 4.6. *Let ν_1, ν_2 be parallel vector fields on the connected surface M and $\nu = \alpha \nu_1 + \beta \nu_2$. Suppose that for every $p \in M$, $\nu_1(p), \nu_2(p)$ are linear independent, then ν is parallel if and only if α and β are constants.*

PROOF. The assumption that ν, ν_1, ν_2 are parallel yields

$$d\alpha \nu_1 + d\beta \nu_2 = 0.$$

But this implies $d\alpha = d\beta = 0$ since ν_1, ν_2 are linear independent. Conversely, it is obvious that if α and β are constants then ν is parallel. □

Among all hyperspheres $HP(\mathbf{n}, c) \cap H_+^n(0, R)$ (\mathbf{n} is timelike) of the hyperbolic space $H_+^n(0, R)$, the case of right hyperspheres, i.e. $\mathbf{n} = (0, 0, \dots, 1)$, are special. The following theorem give some necessary and sufficient conditions for a surface lying in a hyperbolic space to be a part of a right hypersphere.

THEOREM 4.7. *Let M be a surface contained in $H_+^n(0, R)$. The following statements are equivalent:*

1. M is contained in a right hypersphere;
2. \mathbf{n}_r^* is parallel for any $r > 0$;
3. there exists two different parallel normal fields $\mathbf{n}_{r_1}^*, \mathbf{n}_{r_2}^*$;
4. there exists $r > 0$, such that $A^{\mathbf{n}_r^*} = -\alpha \text{id}$, where α is constant.

PROOF. (1. \Rightarrow 2.) It is not hard to see that, because $M \subset \{x_{n+1} = c\} \cap H_+^n(0, R)$, for every $r > 0$,

$$(5) \quad \mathbf{n}_r^* = \alpha \mathbf{X} + \beta \mathbf{v},$$

where α, β are constants. Since \mathbf{X} is parallel and $\mathbf{v} = (0, 0, \dots, 0, -1)$ is constant, \mathbf{n}_r^* is parallel.

(2. \Rightarrow 3.) Obviously.

(3. \Rightarrow 1.) Because \mathbf{X} is a parallel normal field and $\{\mathbf{n}_{r_1}^*, \mathbf{n}_{r_2}^*\}$ is a basis of $N_p M$, we have the linear expression

$$(6) \quad \mathbf{X} = \alpha \mathbf{n}_{r_1}^* + \beta \mathbf{n}_{r_2}^*,$$

where α, β are constants by virtue of Lemma 4.6. Since the last coordinates of $\mathbf{n}_{r_1}^*$ and $\mathbf{n}_{r_2}^*$ are constants, the last coordinate of \mathbf{X} is constant.

(1. \Rightarrow 4.) The equation (5) implies that

$$A^{\mathbf{n}_r^*} = -\alpha \text{id}.$$

(4. \Rightarrow 1.) By the assumption, M is \mathbf{n}_r^* -umbilic. By virtue of Theorem 4.5, $M \subset HP(\mathbf{a}, c)$, where \mathbf{a} is a unit vector. Except at most one point, where \mathbf{X} is parallel to \mathbf{a} ,

$$\mathbf{n}_r^* = \alpha \mathbf{X} + \beta \mathbf{a},$$

where β is a differential function on M .

Since $\langle \mathbf{n}_r^*, \mathbf{n}_r^* \rangle = 2r$, $\langle \mathbf{X}, \mathbf{X} \rangle = -R^2$, $\langle \mathbf{X}, \mathbf{a} \rangle = c$ we obtain the following equation

$$2r = -\alpha^2 R^2 + 2\alpha c \beta + \mathbf{a}^2 \beta^2.$$

Thus, β is constant and therefore so is the last coordinate of \mathbf{X} . □

The following theorem give another necessary and sufficient condition for a surface to be a part of a right hypersphere of a hyperbolic space without the assumption of lying in the hyperbolic space.

THEOREM 4.8. *Let M be a surface in \mathbb{R}_1^{n+1} . The following statements are equivalent*

1. *there exists $r > 0$ such that \mathbf{n}_r^* is parallel, not constant, and M is \mathbf{n}_r^* -umbilic;*
2. *M is contained in a right hypersphere in a hyperbolic space.*

PROOF. ((1) \Rightarrow (2)) Since M is \mathbf{n}_r^* -umbilic, \mathbf{n}_r^* is parallel and $\langle \mathbf{n}_r^*, \mathbf{n}_r^* \rangle = 2r$; $d\mathbf{n}_r^* = \alpha d\mathbf{X}$, $\alpha = \text{const.} \neq 0$, by virtue of Lemma 4.2 in [5]. Therefore,

$$(7) \quad \mathbf{n}_r^* = \alpha \mathbf{X} + \mathbf{a},$$

where \mathbf{a} is constant.

Let $\mathbf{v} = (0, 0, \dots, 0, -1)$. From (7) we have

$$\mathbf{X} - \frac{1}{\alpha}(\mathbf{v} - \mathbf{a}) = \frac{1}{\alpha}(\mathbf{n}_r^* - \mathbf{v}).$$

A simple calculation yields

$$\langle \mathbf{X} - \frac{1}{\alpha}(\mathbf{v} - \mathbf{a}), \mathbf{X} - \frac{1}{\alpha}(\mathbf{v} - \mathbf{a}) \rangle = -\frac{1}{\alpha^2},$$

i.e. M is contained in the hyperbolic space with center $\frac{1}{\alpha}(\mathbf{v} - \mathbf{a})$ and radius $R = \frac{1}{\alpha}$, and hence contained in a right hypersphere by virtue of Theorem 4.7.

((2) \Rightarrow (1)) is obvious by Theorem 4.7. □

The following is somewhat similar to the first statement of Lemma 4.2 in [5].

THEOREM 4.9. *Let M be a connected surface in \mathbb{R}_1^{n+1} . If there exists $r > 0$, such that M is n_r^* -umbilic and for every $i, j \in \{1, 2, \dots, n-1\}$*

$$(8) \quad [(\mathbf{n}_r^*)_{u_i}^T]_{u_j} = [(\mathbf{n}_r^*)_{u_j}^T]_{u_i}$$

then $A_p^{n_r^} = -\alpha id$, where α is constant.*

PROOF. By the assumption, we have

$$(\mathbf{n}_r^+)_{u_i}^T = \alpha \mathbf{X}_{u_i}, \quad i = 1, 2, \dots, n-1.$$

Therefore, for every $i, j \in \{1, 2, \dots, n-1\}$

$$[(\mathbf{n}_r^*)_{u_i}^T]_{u_j} = \alpha_{u_j} \mathbf{X}_{u_i} + \alpha \mathbf{X}_{u_i u_j},$$

and

$$[(\mathbf{n}_r^*)_{u_j}^T]_{u_i} = \alpha_{u_i} \mathbf{X}_{u_j} + \alpha \mathbf{X}_{u_j u_i}.$$

Since $[(\mathbf{n}_r^*)_{u_i}^T]_{u_j} = [(\mathbf{n}_r^*)_{u_j}^T]_{u_i}$ and $\mathbf{X}_{u_i u_j} = \mathbf{X}_{u_j u_i}$, we have

$$\alpha_{u_i} \mathbf{X}_{u_j} - \alpha_{u_j} \mathbf{X}_{u_i} = 0;$$

and hence $\alpha_{u_i} = \alpha_{u_j} = 0$ because $\mathbf{X}_{u_i}, \mathbf{X}_{u_j}$ are linear independent; and therefore α is constant because M is connected. \square

5 Examples

We construct some concrete examples to illustrate the above results.

EXAMPLE 5.1. This example shows that there exists an \mathbf{n}_r^* -umbilic surface but not totally umbilic.

Let M be a parametric surface in R_1^4 , defined by the parametric equation

$$\mathbf{X}(u, v) = \left(\frac{1}{2}u^2, au - \frac{1}{2}u^2, u^2 + v^2, u \right), \quad v > 0, \quad u > 1, \quad a = \sqrt{3} - 1$$

A direct computation shows that \mathbf{X} is spacelike and

$$\mathbf{n}_a^- = (1, 1, 0, a),$$

$$\mathbf{n}_a^+ = \left(\frac{-a^2 + 4ua - 2u^2}{a^2 - 2ua + 2u^2}, \frac{a^2 - 2u^2}{a^2 - 2ua + 2u^2}, 0, a \right).$$

Since \mathbf{n}_a^- is constant, M is \mathbf{n}_a^- -flat. We can check that $\mathbf{X} \subset HP(\mathbf{n}_a^-, 0)$.

Calculating the first and the second fundamental forms (with respect to \mathbf{n}_a^+) yields

$$(g_{ij}) = \begin{pmatrix} 6u^2 - 2au + a^2 - 1 & 4uv \\ 4uv & 4v^2 \end{pmatrix},$$

and

$$(b_{ij}(\mathbf{n}_a^+)) = \begin{pmatrix} \frac{-2a^2 + 4au}{a^2 - 2au + 2u^2} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, the principal curvatures $k_1^{\mathbf{n}_a^-}$ and $k_2^{\mathbf{n}_a^-}$ are the solutions of the following equation

$$4v^2 (2u^2 - 2au + a^2 - 1) k^2 - 4v^2 \left(\frac{-2a^2 + 4au}{a^2 - 2au + 2u^2} \right) k = 0.$$

It is easy to see that $k_1^{\mathbf{n}_a^+} = 0$ and $k_2^{\mathbf{n}_a^+} \neq 0$. Thus, M is not \mathbf{n}_a^+ -umbilic.

EXAMPLE 5.2. This is an example of a totally umbilic surfaces, but the curvature λ is not constant.

Consider the equidistance hypersurface in $H_+^3(0, 1)$

$$M = H_+^3(0, 1) \cap \{x_1 = 0\} = \mathbf{X}(\mathbb{R}^2)$$

defined by

$$\mathbf{X}(u, v) = (0, u, v, \sqrt{u^2 + v^2 + 1}); \quad (u, v) \in \mathbb{R}^2.$$

A direct computation yields

$$\begin{aligned} \mathbf{X}_u &= \left(0, 1, 0, \frac{u}{\sqrt{u^2 + v^2 + 1}}\right), \quad \mathbf{X}_v = \left(0, 0, 1, \frac{v}{\sqrt{u^2 + v^2 + 1}}\right); \\ g_{11} &= \frac{v^2 + 1}{u^2 + v^2 + 1}, \quad g_{12} = g_{21} = \frac{-uv}{u^2 + v^2 + 1}, \quad g_{22} = \frac{u^2 + 1}{u^2 + v^2 + 1}; \\ \mathbf{n}_r^\pm &= \left(\pm \sqrt{\frac{r^2}{u^2 + v^2 + 1}} + 2r, \frac{ru}{\sqrt{u^2 + v^2 + 1}}, \frac{rv}{\sqrt{u^2 + v^2 + 1}}, r\right); \\ \mathbf{X}_{uu} &= \left(0, 0, 0, \frac{v^2 + 1}{(u^2 + v^2 + 1)^{3/2}}\right), \quad \mathbf{X}_{vv} = \left(0, 0, 0, \frac{u^2 + 1}{(u^2 + v^2 + 1)^{3/2}}\right), \\ \mathbf{X}_{uv} &= \left(0, 0, 0, \frac{-uv}{(u^2 + v^2 + 1)^{3/2}}\right); \\ (g_{ij}) &= \frac{1}{u^2 + v^2 + 1} \begin{pmatrix} v^2 + 1 & -uv \\ -uv & u^2 + 1 \end{pmatrix}; \quad (g_{ij})^{-1} = \begin{pmatrix} u^2 + 1 & uv \\ uv & v^2 + 1 \end{pmatrix}; \\ (b_{ij}^\pm) &= \frac{-r}{(u^2 + v^2 + 1)^{3/2}} \begin{pmatrix} v^2 + 1 & -uv \\ -uv & u^2 + 1 \end{pmatrix}; \\ (a_{ij}^\pm) &= (b_{ij}^\pm)(g_{ij})^{-1} = \frac{-r}{\sqrt{u^2 + v^2 + 1}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \\ [(\mathbf{n}_r^+)_u^T]_v &= \left(0, \frac{-rv}{\sqrt{(u^2 + v^2 + 1)^3}}, 0, \frac{-2ruv}{(u^2 + v^2 + 1)^2}\right); \\ [(\mathbf{n}_r^+)_v^T]_u &= \left(0, 0, \frac{-ru}{\sqrt{(u^2 + v^2 + 1)^3}}, \frac{-2ruv}{(u^2 + v^2 + 1)^2}\right). \end{aligned}$$

We can see that M is totally umbilic. Moreover,

$$k_p^{\mathbf{n}_r^\pm} = \frac{-r}{\sqrt{u^2 + v^2 + 1}}$$

is not constant and $[(\mathbf{n}_r^+)_u^T]_v \neq [(\mathbf{n}_r^+)_v^T]_u$ (see Theorem 4.9).

EXAMPLE 5.3. This is an example of a ν -umbilic but neither \mathbf{n}_r^+ - nor \mathbf{n}_r^- -umbilic for any $r \in \mathbb{R}_+$. Let

$$\mathbf{X} : (0, \frac{\pi}{2}) \times (-\frac{\pi}{2}, 0) \rightarrow \mathbb{R}_1^4, \quad (u, v) \mapsto (u, \sin v, v, \cos u).$$

A direct computation yields

$$\begin{aligned} \mathbf{X}_u &= (1, 0, 0, -\sin u), \quad \mathbf{X}_v = (0, \cos v, 1, 0); \\ \mathbf{X}_{uu} &= (0, 0, 0, -\cos u), \quad \mathbf{X}_{uv} = \mathbf{X}_{vu} = (0, 0, 0, 0), \quad \mathbf{X}_{vv} = (0, -\sin v, 0, 0); \\ g_{11} &= \langle \mathbf{X}_u, \mathbf{X}_u \rangle = \cos^2 u > 0, \quad g_{12} = \langle \mathbf{X}_u, \mathbf{X}_v \rangle = 0, \quad g_{22} = \langle \mathbf{X}_v, \mathbf{X}_v \rangle = 1 + \cos^2 v > 0; \end{aligned}$$

$$\begin{aligned} \mathbf{n}_r^+ &= \left(-r \sin u, -\sqrt{\frac{r^2 \cos^2 u + 2r}{1 + \cos^2 v}}, \cos v \sqrt{\frac{r^2 \cos^2 u + 2r}{1 + \cos^2 v}}, r \right); \\ \mathbf{n}_r^- &= \left(-r \sin u, \sqrt{\frac{r^2 \cos^2 u + 2r}{1 + \cos^2 v}}, -\cos v \sqrt{\frac{r^2 \cos^2 u + 2r}{1 + \cos^2 v}}, r \right); \end{aligned}$$

$$(b_{ij}^{\mathbf{n}_r^\pm}) = \begin{pmatrix} r \cos u & 0 \\ 0 & \mp \sin v \sqrt{\frac{r^2 \cos^2 u + 2r}{1 + \cos^2 v}} \end{pmatrix};$$

$$(g_{ij}) = \begin{pmatrix} \cos^2 u & 0 \\ 0 & 1 + \cos^2 v \end{pmatrix};$$

$$(9) \quad (a_{ij}^{\mathbf{n}_r^\pm}) = (b_{ij}^{\mathbf{n}_r^\pm}) \cdot (g_{ij})^{-1} = \begin{pmatrix} \frac{r}{\cos u} & 0 \\ 0 & \mp \sin v \sqrt{\frac{r^2 \cos^2 u + 2r}{(1 + \cos^2 v)^3}} \end{pmatrix};$$

$$(10) \quad k_1^{\mathbf{n}_r^\pm}(P) = \frac{r}{\cos u}, \quad k_2^{\mathbf{n}_r^\pm}(p) = \mp \sin v \sqrt{\frac{r^2 \cos^2 u + 2r}{(1 + \cos^2 v)^3}}.$$

At each point $p = x(u, v) \in M$, let $\nu(p) = \mathbf{n}_{r_p}$, where $r_p = \frac{2 \sin^2 v \cos^2 u}{(1 + \cos^2 v)^3 - \cos^4 u \sin^2 v}$. We can see that ν is a smooth normal vector field on M and M is ν -umbilic but neither \mathbf{n}_r^+ - nor \mathbf{n}_r^- -umbilic for any $r \in \mathbb{R}_+$.

EXAMPLE 5.4. Let

$$\mathbf{X}(u, v) = \left(u, \sin v, \cos v, \sqrt{2 + u^2} \right), \quad u \in \mathbb{R}, \quad v \in (-\pi/2, \pi/2).$$

Because $\langle \mathbf{X}, \mathbf{X} \rangle = -1$, $M \subset H_+^4(0, 1)$. A direct computation yields

$$\mathbf{X}_u = \left(1, 0, 0, \frac{u}{\sqrt{u^2 + 2}} \right), \quad \mathbf{X}_v = (0, \cos v, -\sin v, 0);$$

$$\begin{aligned}
g_{11} &= \frac{2}{2+u^2}, \quad g_{12} = g_{21} = 0, \quad g_{22} = 1; \\
\mathbf{n}_r^\pm &= \left(\frac{ru}{\sqrt{u^2+2}}, \pm \sin v \sqrt{-\frac{u^2 r^2}{u^2+2} + r^2 + 2r}, \pm \cos v \sqrt{-\frac{u^2 r^2}{u^2+2} + r^2 + 2r}, r \right); \\
\mathbf{X}_{uu} &= \left(0, 0, 0, \frac{2}{(u^2+2)^{3/2}} \right), \quad \mathbf{X}_{uv} = \mathbf{X}_{vu} = (0, 0, 0, 0), \quad \mathbf{X}_{vv} = (0, -\sin v, -\cos v, 0) \\
b_{11}^{\mathbf{n}_r^\pm} &= \frac{-2r}{(u^2+2)^{3/2}}, \quad b_{12}^{\mathbf{n}_r^\pm} = 0, \quad b_{22}^{\mathbf{n}_r^\pm} = \mp \sqrt{\frac{2r(u^2+r+2)}{u^2+2}}; \\
k_1^{\mathbf{n}_r^\pm} &= \frac{-r}{\sqrt{u^2+2}}, \quad k_2^{\mathbf{n}_r^\pm} = \mp \sqrt{\frac{2r(u^2+r+2)}{u^2+2}}.
\end{aligned}$$

We can see that $k_1^{\mathbf{n}_r^+} > k_2^{\mathbf{n}_r^+}$ while $k_1^{\mathbf{n}_r^-} < k_2^{\mathbf{n}_r^-}$ for any $r > 0$. Thus, M is not ν -umbilic, for any normal vector field $\nu \neq \mathbf{X}$.

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DANG VAN CUONG
HUE GEOMETRY GROUP
DEPARTEMENT OF MATHEMATICS
DUY TAN UNIVERSITY
DANANG
VIETNAM

E-mail address: cuongdangvan@gmail.com

DOAN THE HIEU
HUE GEOMETRY GROUP
DEPARTEMENT OF MATHEMATICS
COLLEGE OF EDUCATION, HUE UNIVERSITY
HUE
VIETNAM

E-mail address: dthehieu@yahoo.com